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OLYMPIAD PROBLEMS FROM ALL OVER THE WORLD

VOLUME 5
9th GRADE CONTENT

Cartea Românească
EDUCAȚIONAL

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Chapter I
Problems

1. Let α be a fixed real number. Determine all functions $f : \mathbb{R} \to \mathbb{R}$ satisfying $f(f(x+y)\mathrm{f}(x-y)) = x^2 + \alpha y f(y)$ for all $x, y \in \mathbb{R}$.

WALTER JANOUS, AUSTRIAN NMO, 2017

2. Let M be a set of 2017 positive integers. For every non-empty $A \subset M$, we define $f(A) = \{x \in M : x \text{ is divisible by odd number of elements of } A\}$.

Find the minimum number of colors such that it is possible to paint all non-empty subset of M in such a way that, whenever $A \neq f(A)$, the sets A and $f(A)$ are in different colors.

ALEKSANDAR IVANOV, BULGARIAN NMO, 2017

3. Let n be a positive integer and $a_1, a_2, \ldots, a_{2n}$ be $2n$ distinct integers. Given that the equation $|x - a_1|\,|x - a_2| \ldots |x - a_{2n}| = (n!)^2$ has an integer solution $x = m$, find m in terms of $a_1, \ldots, a_{2n}$.

SINGAPORE, SMO, 2017

4. We consider the real numbers x, y and z, which satisfy the system of equations:

$$\begin{cases} x + y + z = 3 \\ xy + yz + xz = 2 \end{cases}.$$

Compute $\max(x) + \min(x)$.

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5. Given 7 distinct positive integers, prove that there is an infinite arithmetic progression of positive integers $a, a + d, a + 2d, \ldots$, with $a \leq d$, that contains exactly 3 or 4 of the 7 given integers.

SINGAPORE, SMO, 2017

6. Solve the equation $x^2(2 - x)^2 = 1 + 2(1 - x)^2$.

FINBAR HOLLAND, IRELAND SHL, 2017

7. Show that, for all x, y, z, w, $(x - w)(y - z) + (y - w)(z - x) + (z - w)(x - y) = 0$ and $\sin(x - w)\sin(y - z) + \sin(y - w)\sin(z - x) + \sin(z - w)\sin(x - y) = 0$.

FINBAR HOLLAND, IRELAND SHL, 2017

8. Let $f(n) = 4n^2 + 7n^2 + 3n + 6$. Prove that if n is an integer, then $f(n)$ is not the cube of an integer.

TOM LAFFEY, IRELAND SHL, 2017

9. For each positive integer n, let $c_p = 2017^n$. Suppose that a function $f : \mathbb{N} \to \mathbb{R}$ satisfies the following two conditions:

(i) $f(m + n) \leq 2017 \cdot f(m) \cdot f(n + 325)$ for each positive integer m, n;

(ii) $0 < f(c_n + 1) < f(c_n)^{2017}$ for each positive integer n.

Show that there exist a sequence $a_1, a_2, ...,$ satisfying the following condition.

For all positive integers n, k with $a_k < n$, we have $f(n)^{c_k} < f(c_k)^n$.

In the problem, $\mathbb{N}$ is the set of positive integers and $\mathbb{R}$ is the set of reels.

KOREAN NMO, 2017

10. A function $f : \mathbb{Z} \to \mathbb{Z}$ is called loggy if it satisfies the following two conditions:

(i) $f(xy) \equiv f(x) + f(y) \pmod 8$ for all $x, y \in \mathbb{Z}$ that are not divisible by 17;

(ii) $f(x + 17) \equiv f(x) \pmod 8$ for all $x \in \mathbb{Z}$.

Determine, with proof:

a) if there exists a loggy function for which $f(2) = 1$;

b) if there exists a loggy function for which $f(3) = 1$.

BERND KREUSSLER, IRELAND NMO, 2017

11. Let n be a positive integer and $a_1, ..., a_n$ positive real numbers. Let:

$$s_k = a_1^k + ... + a_n^k \text{ for } k = 1, 2, 3, ...$$

Prove that $\dfrac{s_5 s_1^3}{5} - \dfrac{s_4 s_2 s_1^2}{4} + \dfrac{s_2^4}{20} \geq 0.$

TOM LAFFEY, IRELAND SHL, 2017

12. Suppose x, y, z are positive numbers that sum to π. Prove that:

$$\frac{\sin 2x + \sin 2y + \sin 2z}{\sin x + \sin y + \sin z} \leq 1,$$

with equality if $x = y = z = \dfrac{\pi}{3}$.

FINBAR HOLLAND, IRELAND SHL, 2017

13. There are some boys and some girls at a party. A set of boys is said to be *sociable* if every girl at the party knows at least one boy in that set, and similarly a set of girls is said to be *sociable* if every boy at the party knows at least one girl in that set.

Suppose that the number of sociable sets of boys is odd. Prove that the number of sociable sets of girls is also odd.

Note: Acquaintance is mutual.

MARK FLANAGAN, IRELAND NMO, 2017

14. Suppose A, B, and C are the angles in an acute-angled triangle. Prove that:

$$\frac{\sin 2A+\sin 2B+\sin 2C}{\cos A+\cos B+\cos C}\leq\sqrt{3}.$$

FINBAR HOLLAND, IRELAND SHL, 2017

15. In ΔABC, the following relationship holds:

$$ab^7+bc^7+ca^7\geq 62208r^8.$$

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16. In ΔABC, the following relationship holds:

$$\prod(m_a+r_a)\geq 8\sqrt{\prod w_a h_a}+\left(\sqrt{\prod m_a}-\sqrt{\prod r_a}\right)^2.$$

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17. In ΔABC, the following relationship holds:

$$\left(\sum\sqrt{m_a}\right)^2\geq\sum m_a+6\sqrt[3]{rs^2}.$$

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18. If in ΔABC, K-Lemoine's point, then the following relationship holds:

$$\sum\sqrt[3]{a}\cdot KA^2\geq\frac{\sqrt[3]{abc}(a\sqrt[3]{a^2}+b\sqrt[3]{b^2}+c\sqrt[3]{c^2})}{\sqrt[3]{a}+\sqrt[3]{b}+\sqrt[3]{c}}.$$

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19. If $a, b, c\geq 0$, then:

$$(\cos 50^\circ+\cos 70^\circ)\sum a^2\geq\frac{2\cos 10^\circ}{1+2\sqrt{3}\cos 10^\circ}\sum(a^2+ab).$$

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20. In ΔABC, the following relationship holds:

$$a^2r_a+b^2r_b+c^2r_c\geq 108r^3.$$

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21. In any triangle ABC, the following relationship holds:

$$\sum a^2c^2\sin 2B+\sum b^2c^2\sin 2A\geq 432\sqrt{3}r^4.$$

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22. In ΔABC, the following relationship holds:

$$(a\cot 20^\circ + b\cot 40^\circ + c\cot 80^\circ)^3 > 9\sqrt{3}\left(\frac{a^3}{r_a}+\frac{b^3}{r_b}+\frac{c^3}{r_c}\right).$$

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23. If $a, b \in \mathbb{N} \setminus \{0\}$, then:

$$\left(a+\sqrt{ab}+b\right)\left(\frac{1}{a}+\frac{1}{\sqrt{ab}}+\frac{1}{b}\right)^{ab} \le \left(\frac{3a+3b}{1+ab}\right)^{1+ab}.$$

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24. If $a, b, x, y, z > 0$, then:

$$\sqrt[3]{\left(a+\frac{b(x+y+z)}{x}\right)\left(a+\frac{b(x+y+z)}{y}\right)\left(a+\frac{c(x+y+z)}{z}\right)} \ge a+3b.$$

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25. If in ΔABC $2Rr = 1$, then:

$$16R^2 + 12r^2 \ge 19.$$

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26. If in ΔABC $\tan\frac{B}{2}\tan\frac{C}{2}=\frac{1}{3}$, then:

$$s^2 \ge 27R^2\sin^2\frac{A}{2}.$$

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27. If $a, b, c > 0$, then:

$$\left(6\sqrt[3]{\frac{a}{b}}-\frac{a^2}{b^2}\right)+\left(6\sqrt[3]{\frac{b}{c}}-\frac{b^2}{c^2}\right)+\left(6\sqrt[3]{\frac{c}{a}}-\frac{c^2}{a^2}\right)\le 15.$$

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28. In ΔABC, the following relationship holds:

$$\sum\frac{1}{(a+2\sqrt{ab})(b+2\sqrt{ab})}\le\frac{1}{18Rr}.$$

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29. If in ΔABC:

$$x=\cos^{-1}\left(\frac{a}{b+c}\right),\ y=\cos^{-1}\left(\frac{b}{c+a}\right),\ z=\cos^{-1}\left(\frac{c}{a+b}\right),\ \text{then:}$$

$$\tan\frac{x}{2}\tan\frac{y}{2}\tan\frac{z}{2}\le\frac{\sqrt{3}}{9}.$$

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30. In ΔABC, the following relationship holds:

$$(a^2+b^2+c^2)\sqrt{a^2+b^2+c^2}\ge 6abc\sqrt{6\cos A\cos B\cos C}.$$

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31. Let be $n\in\mathbb{N}^*\setminus\{1\}$ și $a_k\in\mathbb{R}$, $k\in\overline{1,n}$. Prove that:

$$\sum_{k=1}^{n}\sqrt{a_k^2-a_ka_{k+1}+a_{k+1}^2}\ge\sum_{k=1}^{n}a_k;\ a_{n+1}=a_1.$$

D.M. BĂTINEȚU-GIURGIU, NECULAI STANCIU,
ROMANIA, RMM SUMMER EDITION, 2016

32. Prove that if $a, b, c, d > 0$, then:

$$a^2+b^2+c^2+d^2=1;\ abc+bcd+cda+dab=\frac{1}{2}.$$

$$\sum\frac{a^2}{1+2bcd}\ge\frac{4}{5}.$$

DANIEL SITARU, ROMANIA, RMM SUMMER EDITION, 2016

33. If $x, y, z\in(0,\infty)$, then: $x+y+z+\dfrac{1}{x^3}+\dfrac{1}{y^3}+\dfrac{1}{z^3}\ge\dfrac{12}{\sqrt{3\sqrt{3}}}.$

DANIEL SITARU, ROMANIA, RMM SUMMER EDITION, 2016

34. We consider $f(x)=x^3+bx^2+cx+d$ with $f(2014)=2013$ and $g(x)=x^2-2x+2014$ such that the equation $f(g(x))=0$ doesn't have real roots. Solve the equation $f(x)=0$ knowing that it has three distinct natural numbers roots.

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35. If $a, b, c > 0$, $(a+b)(b+c)(c+a)=a^2b^2c^2$, then:

$$\sqrt{a+b}+\sqrt{b+c}+\sqrt{c+a}\ge 6.$$

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36. If $a, b, c > 0$, $a^2 + b^2 + c^2 = 3$, then:

$$6\left(\frac{b^2}{\sqrt{a^2+3}}+\frac{c^2}{\sqrt{b^2+3}}+\frac{a^2}{\sqrt{c^2+3}}\right)\geq (a+b+c)^2.$$

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37. In ΔABC, the following relationship holds:

$$\frac{a^4}{\tan 22^\circ}+\frac{b^2}{\tan 22^\circ \tan 23^\circ}+\frac{c^4}{\tan 23^\circ}>48S^2.$$

DANIEL SITARU, RMM, ROMANIA

38. There are 12 chairs which are aligned and labeled by numbers 1; 2; ...; 12 from left to right. A grasshopper can jump from one chair to another following the rule: from a chair with number k it can jump to the chair with number n if and only if $|k - n| = 5$ or $|k - n| = 8$. It is known that grasshopper managed to do the jump so that it visited all chairs exactly once. What chair could be the initial position for the grasshopper?

UKRAINIAN NMO, 2016

39. Let x, y, z be real numbers from segment [0; 1]. Prove that:

$$(x^4 + y^4 + z^4) + (x^5 + y^5 + z^5) + (x - y)^6 + (y - z)^6 + (z - x)^6 \leq 6.$$

YASINSKII VYACHESLAV, UKRAINIAN NMO, 2016

40. Find all real numbers x satisfying the following equation:

$$(x + \{x\})^2 - (x + \{x\}) = 6[x]\{x\} - 1,$$

where $[x]$ and $\{x\}$ denote the integer part and fractional part of x, respectively.

NGUYEN VIET HUNG, VIETNAM, RMM AUTUMN EDITION, 2016

41. A convex quadrilateral $ABCD$ is inscribed in a circle. The lines AD and BC meet at point E. Points M and N are taken on the sides AD and BC, respectively, so that $AM : MD = BN : NC$. Let the circumcircles of triangle EMN and quadrilateral $ABCD$ intersect at point X and Y. Prove that either the lines AB, CD and XY have a common point, or they are all parallel.

DUŠAN ĐJUKIĆ, SERBIAN NMO, 2017

42. Let p be a prime. Show that $\sqrt[3]{p}+\sqrt[3]{p^5}$ is irrational.

THAILAND NMO, 2017

43. Find all functions $f : \mathbb{R} \to \mathbb{R}$ which satisfy:

$$f(f(x) - y) \leq xf(x) + f(y) \ (1)$$

for all real numbers x and y.

THAILAND NMO, 2017

44. Find all functions $f: \mathbb{Q}^+ \to \mathbb{Q}^+$ such that:

$$f(xf(x) + f(y)) = (f(x))^2 + y \quad (1)$$

for all positive rational x, y.

THAILAND NMO, 2017

45. Find all functions $f: \mathbb{N} \to \mathbb{N}$ such that for every positive integer m the following is true: If we denote by $d_1, d_2, \ldots, d_n$ all the divisors of number m, then:

$$f(d_1) \cdot f(d_2) \cdot \ldots \cdot f(d_n) = m.$$

PAVEL CALABEK, CZECH & SLOVAK NMO, 2017

46. Find all triplets of integers (a, b, c) such that each of the fractions:

$$\frac{a}{b+c}, \frac{b}{c+a}, \frac{c}{a+b} \text{ is an integer.}$$

JAROSLAV SVRCEK, CZECH & SLOVAK NMO, 2017

47. Let ABC be an acute triangle with altitude AD. The bisectors of angles BAD, CAD intersect side BC at E, F, respectively. The circumcircle of triangle AEF intersects sides AB, AC at G, H, respectively. Prove that lines EH, FG, and AD pass through a common point.

PATRIK BAK, CZECH & SLOVAK NMO, 2017

48. Let be $f: \mathbb{R} \to \mathbb{R}$, $f(x) = \sum_{k=0}^{n} a_k x^k$, where $a_k \geq 0$, $\forall\ k = \overline{0,n}$. If $f(4) = 8$ and $f(9) = 18$, then find $\max(f(6))$ and the value for which this maximum is achieved.

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49. If $a, b, c > 0$, $a + b + c = 3$, $x \in \mathbb{R}$, then:

$$\left(\sqrt[3]{a\sin^2 x} + \sqrt[3]{b\cos^2 x}\right)\left(\sqrt[3]{b\sin^2 x} + \sqrt[3]{c\cos^2 c}\right)\left(\sqrt[3]{c\sin^2 x} + \sqrt[3]{a\cos^2 x}\right) \leq 4$$

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50. If $x, y, z > 0$, then:

$$\sum \frac{x^3}{y^3} + 2\sum \frac{y}{x} + 2\sum \frac{x}{y} + \sum \frac{y^3}{x^3} \geq \sum \frac{x^2}{y^2} + 12 + \sum \frac{y^2}{x^2}.$$

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51. If $x, y, z > 0$, $x + y + z = 3$, then:

$$\left(\sqrt[3]{x} + \sqrt[3]{y}\right)^3 + \left(\sqrt[3]{y} + \sqrt[3]{z}\right)^3 + \left(\sqrt[3]{z} + \sqrt[3]{x}\right)^3 \leq 24.$$

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52. If in ΔABC $S = 2$, then:

$$\frac{(2s-a)(2s-b)(2s-c)}{(2+a\sqrt{\sin A})(2+b\sqrt{\sin B})(2+c\sqrt{\sin C})} \geq \frac{1}{\sqrt{\sin A \sin B \sin C}}.$$

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53. In acute ΔABC, the following relationship holds:

$$a^2b^2c^2 \sin 2A \sin 2B \sin 2C \cos A \cos B \cos C \leq S^3.$$

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54. In ΔABC, the following relationship holds:

$$(am_a + bm_b + cm_c)(am_a^3 + bm_b^3 + cm_c^3) \leq (a+b+c)(am_a^4 + bm_b^4 + cm_c^4).$$

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55. If $a, b, c > 0$, then:

$$\frac{\sum \sqrt[3]{(a+3b)(2a+2b)(3a+b)}}{\sqrt{ab}+\sqrt{bc}+\sqrt{ca}} \geq 4.$$

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56. If $a, b, c > 0$, $a^2 + b^2 + c^2 = 3$, then:

$$\frac{a^3+1}{\sqrt{a^2-a+1}} + \frac{b^3+1}{\sqrt{b^2-b+1}} + \frac{c^3+1}{\sqrt{c^2-c+1}} \geq 6.$$

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57. In ΔABC, the following relationship holds:

$$\frac{(a+b)^4}{ab} \geq \frac{64s^2}{3}.$$

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58. If $a, b, c, d > 0$, $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}=1$, then:

$$\sqrt[3]{a}+\sqrt[3]{b}+\sqrt[3]{c}+\sqrt[3]{d} \leq \sqrt[3]{abcd}.$$

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59. If $x, y, z \geq 1$, then:

$$\sum \frac{1}{1+\sqrt{(x-1)(y-1)}} \leq \frac{3}{\sqrt[3]{xyz}}.$$

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60. In ΔABC, $\measuredangle B > \measuredangle C$. Let D be the point on side BC such that $\measuredangle DAC = \frac{B-C}{2}$. The circumcircle of ΔACD meets side AB again at E. The circumcircle of ΔABD meets side AC again at F. The internal angle bisector of $\measuredangle BDE$ meets side AB at P. The internal angle bisector of $\measuredangle CDE$ meets side AC at Q. Prove that PQ and AB are perpendicular.

HONG KONG, PREIMO 2017, MOCK EXAM

61. Let a, b, c, d be positive real numbers satisfying $abcd = 1$. Prove that:

$$(a^2b + b^2c + c^2d + d^2a)(ab^2 + bc^2 + cd^2 + da^2) \geq (a + c)(b + d)(ac + bd + 2).$$

When does equality hold?

HONG KONG, PREIMO 2017, MOCK EXAM

62. Prove the following inequality:

$[(x+y)(y+z)(z+x)]^4 \geq \frac{16^3}{27}(x+y+z)^3x^3y^3z^3$, where x, y, z are positive real numbers.

ANDREI BOGDAN UNGUREANU, RMM WINTER EDITION, 2016

63. Prove that if $a, b, c, d \in (0, \infty)$; $\sqrt{3}(ad - bc) = ac + bd \neq 0$, then:

$$d(a + b\sqrt{3}) - c(b - a\sqrt{3}) > 4\sqrt[4]{abcd}.$$

DANIEL SITARU, RMM WINTER EDITION, 2016

64. Prove that in an ABC acute-angled triangle the following relationship holds:

$$\cos\left(\frac{\pi}{4} - A\right) + \cos\left(\frac{\pi}{4} - B\right) + \cos\left(\frac{\pi}{4} - C\right) > \frac{2S}{R^2}.$$

DANIEL SITARU, RMM WINTER EDITION, 2017

65. Prove that in ΔABC:

$$\sum \frac{a^2(b^2 + c^2 - a^2)^3}{b^2c^2} \geq 64S^2(1 - \cos^2 A - \cos^2 B - \cos^2 C).$$

DANIEL SITARU, RMM WINTER EDITION, 2016

66. Let $x_1, x_2, ..., x_n$ be positive real numbers such that $\frac{1}{x_1} + \frac{2}{x_2} + ... + \frac{n}{x_n} = \frac{n(n+1)}{2}$.

Find the minimum possible value of $x_1 + x_2^2 + ... + x_n^n$.

NGUYEN VIET HUNG, RMM SPRING EDITION, 2017

67. Prove that for all $x \in \mathbb{R}$ we have $\cos(\sin x) > |\sin(\cos x)|$.

ABDALLAH EL FARISSI, RMM SPRING EDITION, 2017

68. Let $a, b \in \mathbb{R}$ such that $a + b > 0$, then:

$$\left(\frac{a+b}{2}\right)^n \leq \frac{1}{n+1}\sum_{k=0}^{n} a^k b^{n-k} \leq \frac{a^n + b^n}{2}.$$

ABDALLAH EL FARISSI, RMM SPRING EDITION, 2017

69. Call a function $f : \mathbb{N} \to \mathbb{N}$ lively if $f(a + b - 1) = \underbrace{f(f(...)f(b)...))}_{a \text{ times}}$ for all $a, b \in \mathbb{N}$.

Suppose that g is a lively function such that $g(A + 2018) = g(A) + 1$ holds for some $A \geq 2$.

a) Prove that $g(n + 2017^{2017}) = g(n)$ for all $n \geq A + 2$.

b) If $g(A + 2017^{2017}) \neq g(A)$, determine $g(n)$ for $n \leq A - 1$.

MARKO RADOVANOVIĆ, SERBIAN TST, 2017

70. A $n \times n$ square is divided into unit squares. One needs to place a number of isosceles right triangles with hypotenuse 2, with vertices at grid points, in such a way that every side of every unit square belongs to exactly on triangle (i.e. lies inside it or on its boundary). Determine all numbers n for which this is possible.

DUŠAN ĐJUKIĆ, SERBIAN TST, 2017

71. Let $ABCD$ be a convex quadrilateral with $AC \perp BD$. Prove that there exist points P, Q, R, S on AB, BC, CD, DA, respectively, such that $PR \perp QS$ and the area of quadrilateral $PQRS$ is exactly half that of $ABCD$.

THAILAND TST, 2017

72. Let a and b be real numbers such that $a + b = 1$. Prove the following inequality:

$$\sqrt{1+5a^2} + 5\sqrt{2+b^2} \geq 9.$$

B. BATTSENGEL, MONGOLIAN NMO, 2017

73. Let $ABCD$ be an isosceles trapezoid with $AD = BC$ and $AB \parallel CD$. Let O be the intersection of the diagonals and let M be the midpoint of AD. Circumcircle of BCM intersects AD again at K. Prove that OK is parallel to AB.

B. BAT-OD, MONGOLIAN NMO, 2017

74. The altitudes AD and BE of acute triangle ABC intersect at H. Let F be the intersection of AB and a line that is parallel to the side BC and goes through the circumcenter of ABC. Let M be the midpoint of AH. Prove that $\measuredangle CMF = 90°$.

G. BATZAYA, MONGOLIAN NMO, 2017

75. Let $ABCD$ be a cyclic quadrilateral with circumcenter w, and E be the intersection of the diagonals AC and BD. A line passing through E intersects lines AB, BC at P, Q,